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An Efficient Collocation Method for the Numerical Solutions of the Pantograph-Type Volterra Hammerstein Integral Equations and its Convergence Analysis

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Abstract

In this work, we consider a collocation method for solving the pantograph-type Volterra Hammerstein integral equations based on the first kind Chebyshev polynomials. We use the Lagrange interpolating polynomial to approximate the solution. The convergence of the presented method has been analyzed by over estimating for error. Finally, some illustrative examples are given to test the accuracy of the method. The presented method is compared with the Legendre Tau method.

Keywords: Numerical solution, Collocation method, Pantograph-type, Volterra Hammerstein integral equations, Convergence analysis.

1 | Introduction

This paper aims to obtain the exact approximations by collocation method based on the first kind Chebyshev polynomials for solutions of the following pantograph Volterra Hammerstein integral equations.

$$z(s) = g(s) + \int_0^{qs} R(t,s)G\big(t,z(t)\big)dt, \quad 0 < q < 1, \quad s \in [0,1]\,. \tag{1}$$

Where $g(s)$, $R(s,t)$ and $G(t,z(t))$ are smooth functions on their domains. We assume that the function $G(t,z(t))$ satisfies in the Lipschitz condition with respect to the second variable.

These equations appear in many branches of science such as control theory, biology, ecology and etc [1]-[7]. Severral methods have been proposed to solve the integral equations in [8]-[11], [5], [12]-[16], [6], [17]-[26]. Ansari and Mokhtary [27] presented the Legendre Tau method for solving *Eq. (1)* and

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discussed applying the spectral methods to obtain a reliable numerical solution for *Eq. (1)* according to the well-known existence and uniqueness theorems in [27].

Here, we approximate the solution of *Eq. (1)* using the Lagrange interpolating polynomial. We consider a collocation method based on the first kind Chebyshev polynomials for solving *Eq. (1)* and get a non-linear system that can be solved by Newton method to obtain the solutions at the grid pionts. In order to use the theory of orthogonal polynomials, we transform the *Eq. (1)* to $[-1,1]$ and analyze the convergence of the presented method. Some numerical examples prepared to test the efficiency and accuracy of the proposed method. We compare the numerical results of the presented method with the Legendre Tau method in [27].

2 | Algorithm of the Method

The first kind of Chebyshev polynomials $T_N(s)$ are orthogonal at $[-1,1]$ with respect to the weight function $w(s) = (1 - s^2)^{-1/2}$ and are defined by [28]:

$$T_N(s) = \cos\left(N\cos^{-1}(s)\right).$$

All Chebyshev polynomials $T_N(s)$ can be generated by the following recursion relation

$$T_N(s) = 2sT_{N-1}(s) - T_{N-2}(s), \quad N = 2,3, \ldots.$$

With

$$T_0(s) = 1, T_1(s) = s.$$

The Gauss quadrature formula

$$\int_{-1}^{1} f(s)w(s)ds \approx \sum_{k=0}^{N} f(s_k)w_k.$$

is exact for any polynomial of degree $\leq 2N + 1$.

Now, in order to use the theory of orthogonal Chebyshev polynomials, applying the following change of variable

$$\begin{cases} t = \frac{q}{2}\eta + \frac{q}{2}, & -1 \leq \eta \leq \zeta, \quad 0 < q < 1, \\ s = \frac{1}{2}\zeta + \frac{1}{2}, & -1 \leq \zeta \leq 1. \end{cases}$$

For *Eq. (1)*, we have

$$\tilde{z}(\zeta) = \tilde{g}(\zeta) + \int_{-1}^{\zeta} \tilde{R}(\zeta,\eta)\tilde{G}\left(\eta, \tilde{z}(\eta)\right)d\eta, \quad -1 \leq \zeta \leq 1. \tag{2}$$

Where

$$\begin{cases} \tilde{z}(\zeta) = z\left(\frac{1}{2}\zeta + \frac{1}{2}\right), \\ \tilde{g}(\zeta) = g\left(\frac{1}{2}\zeta + \frac{1}{2}\right), \\ \tilde{R}(\zeta,\eta) = \frac{q}{2}R\left(\frac{1}{2}\zeta + \frac{1}{2}, \frac{q}{2}\eta + \frac{q}{2}\right), \\ \tilde{G}\left(\eta, \tilde{z}(\eta)\right) = G\left(\frac{q}{2}\eta + \frac{q}{2}, z\left(\frac{q}{2}\eta + \frac{q}{2}\right)\right). \end{cases}$$

Using the Lagrange interpolating polynomial, we can approximate $\tilde{z}(\zeta)$ as

$$I_N^C\left(\tilde{z}(\zeta)\right)=\sum_{j=0}^{N} l_j(\zeta)\tilde{z}\left(s_j\right). \tag{3}$$

Where

$$l_i(\eta)=\prod_{j=0,j\neq i}^{N}\frac{\eta-s_j}{s_i-s_j},\quad i=0(1)N.$$

And $\left\{s_j=-cos\left(\frac{(2j+1)\pi}{2N+2}\right)\right\}_{j=0}^{N}$ are the Chebyshev Gauss quadrature points. Also, from *Eq. (3)*, we can write

$$\widetilde{G}\left(\zeta,I_N^C\left(\tilde{z}(\zeta)\right)\right)=\sum_{j=0}^{N} l_j(\zeta)\widetilde{G}\left(s_j,\tilde{z}\left(s_j\right)\right). \tag{4}$$

Now, collocating the nodes $s_i,\ i=0(1)N,$ in *Eq. (2)*, we have

$$\tilde{z}(s_i)=\tilde{g}(s_i)+\int_{-1}^{s_i}\widetilde{R}\,(s_i,\eta)\,\widetilde{G}\left(\eta,I_N^C\left(\tilde{z}(\eta)\right)\right)d\eta,\quad i=0(1)N. \tag{5}$$

Solving the non-linear *system (5)* by the Newton method, we get the values of $\tilde{z}\left(s_j\right)$. Substituting these values into *Eq. (3)*, the values of $I_N^C\left(\tilde{z}(\zeta)\right)$ are obtained, for all $\in[-1,1]$.

3 | Convergence Analysis

In this section, using some useful inequalities and lemmas from [29], [30], we will try to determine over estimate for $\|I_N^C\left(\tilde{z}(\zeta)\right)-\tilde{z}(\zeta)\|_{L_w^2(-1,1)}$.

Theorem 1. Assume that $I_N^C\left(\tilde{z}(\zeta)\right)$ is the spectral collocation solution of *Eq. (2)* given by *Eq. (3)*. If the functions $\tilde{g}(\zeta)$, $\tilde{R}(\zeta,\eta)$ and $\tilde{G}\left(\eta,\tilde{z}(\eta)\right)$ are sufficiently smooth and $\tilde{G}\left(\eta,\tilde{z}(\eta)\right)$ satisfies in the Lipschitz condition with respect to the second variable and $e(\zeta)=I_N^C\left(\tilde{z}(\zeta)\right)-\tilde{z}(\zeta)$, then we have

$$\begin{aligned}\|e\|_{L_w^2(-1,1)}\le{}& CN^{-m}\left(|\tilde{g}|_{H_w^{m;N}(-1,1)}+L|\tilde{z}|_{H_w^{m;N}(-1,1)}\right)\\&+CN^{-1}\left(\|\widetilde{G}\|+N^{-m}L|\tilde{z}|_{H_w^{m;N}(-1,1)}\right).\end{aligned} \tag{6}$$

Proof. The *Eq. (5)* can be written as follows

$$\begin{aligned}\tilde{z}(s_i)={}&\tilde{g}(s_i)+\int_{-1}^{s_i}\widetilde{R}\,(s_i,\eta)\,\widetilde{G}\left(\eta,I_N^C\left(\tilde{z}(\eta)\right)\right)d\eta+\int_{-1}^{s_i}\widetilde{R}\,(s_i,\eta)\,\widetilde{G}\left(\eta,\tilde{z}(\eta)\right)d\eta\\&-\int_{-1}^{s_i}\widetilde{R}\,(s_i,\eta)\,\widetilde{G}\left(\eta,\tilde{z}(\eta)\right)d\eta.\end{aligned} \tag{7}$$

Multiplying *Eq. (7)* by $l_j(\zeta)$ and sum up from 0 to N, we have

$$\begin{aligned}I_N^C\left(\tilde{z}(\zeta)\right)={}&I_N^C\left(\tilde{g}(\zeta)\right)\\&+I_N^C\left(\int_{-1}^{\zeta}\widetilde{R}\,(\zeta,\eta)\left(\widetilde{G}\left(\eta,I_N^C\left(\tilde{z}(\eta)\right)\right)-\widetilde{G}\left(\eta,\tilde{z}(\eta)\right)\right)d\eta\right)\\&+I_N^C\left(\int_{-1}^{\zeta}\widetilde{R}\,(\zeta,\eta)\,\widetilde{G}\left(\eta,\tilde{z}(\eta)\right)d\eta\right).\end{aligned} \tag{8}$$

Subtracting *Eq. (8)* from the *Eq. (2)*, we have

$$e(\zeta)=L_2+I_N^C\left(\int_{-1}^{\zeta}\widetilde{R}\,(\zeta,\eta)\left(\widetilde{G}\left(\eta,I_N^C\left(\tilde{z}(\eta)\right)\right)-\widetilde{G}\left(\eta,\tilde{z}(\eta)\right)\right)d\eta\right)+L_3. \tag{9}$$

Where

$$\begin{cases} L_2 = I_N^C\big(\tilde{g}(\zeta)\big) - \tilde{g}(\zeta), \\ L_3 = I_N^C\big(\int_{-1}^{\zeta} \tilde{R}(\zeta,\eta)\, \tilde{G}\big(\eta, \tilde{z}(\eta)\big)\, d\eta\big) - \int_{-1}^{\zeta} \tilde{R}(\zeta,\eta)\, \tilde{G}\big(\eta, \tilde{z}(\eta)\big)\, d\eta. \end{cases}$$

The *Eq. (9)* can be rewritten as follows

$$e(\zeta) = L_2 + \int_{-1}^{\zeta} \tilde{R}(\zeta,\eta) \left(\tilde{G}\left(\eta, I_N^C\big(\tilde{z}(\eta)\big)\right) - \tilde{G}\big(\eta, \tilde{z}(\eta)\big)\right) d\eta \; + L_3 + L_4. \tag{10}$$

Where

$$L_4 = I_N^C\left(\int_{-1}^{\zeta} \tilde{R}(\zeta,\eta) \left(\tilde{G}\left(\eta, I_N^C\big(\tilde{z}(\eta)\big)\right) - \tilde{G}\big(\eta, \tilde{z}(\eta)\big)\right) d\eta\right)$$
$$- \int_{-1}^{\zeta} \tilde{R}(\zeta,\eta) \left(\tilde{G}\left(\eta, I_N^C\big(\tilde{z}(\eta)\big)\right) - \tilde{G}\big(\eta, \tilde{z}(\eta)\big)\right) d\eta.$$

To simplicity, in the following the notation $\|.\|$ has been used instead of $\|.\|_{L^2_{\varpi}(-1,1)}$. Now, from the *Eq. (10),* we have

$$\|e(\zeta)\| \le \|L_2\| + \left\| \int_{-1}^{\zeta} \tilde{R}(\zeta,\eta) \left(\tilde{G}\left(\eta, I_N^C\big(\tilde{z}(\eta)\big)\right) - \tilde{G}\big(\eta, \tilde{z}(\eta)\big)\right) d\eta \right\| + \|L_3\| + \|L_4\|. \tag{11}$$

Now, in order to determine over estimate for error, we should find the over bound for right hand side of the inequality *Eq. (11)* term by term.

Using inequality (5.5.22) from [29], we have

$$\|L_2\| \le CN^{-m} |\tilde{g}|_{H_w^{m;N}(-1,1)}. \tag{12}$$

Where

$$|\tilde{g}|_{H_w^{m;N}(-1,1)} = \Big(\sum_{k=\min(m,N+1)}^{m} \|\tilde{g}^{(k)}\|^2 \Big)^{1/2}. \tag{13}$$

For *Eq. (12)*, using *Eq. (13)* with $m = 1$ and Lemma 3.8 from [30], we have

$$\begin{aligned} \|L_3\| &\le CN^{-1}\left(\left\| \int_{-1}^{\zeta} \frac{\partial\big(\tilde{R}(\zeta,\eta)\big)}{\partial\zeta}\, \tilde{G}\big(\eta, \tilde{z}(\eta)\big) d\eta + \tilde{R}(\zeta,\zeta)\tilde{G}\big(\zeta, \tilde{z}(\zeta)\big) \right\| \right) \\ &\le CN^{-1}\left(\left\| \int_{-1}^{\zeta} \frac{\partial\big(\tilde{R}(\zeta,\eta)\big)}{\partial\zeta}\, \tilde{G}\big(\eta, \tilde{z}(\eta)\big) d\eta \right\| + \|\tilde{R}(\zeta,\zeta)\| \|\tilde{G}\big(\zeta, \tilde{z}(\zeta)\big)\| \right) \\ &\le CN^{-1}\Big(C\|\tilde{G}\big(\eta, \tilde{z}(\eta)\big)\| + \|\tilde{R}(\zeta,\zeta)\| \|\tilde{G}\big(\zeta, \tilde{z}(\zeta)\big)\| \Big). \end{aligned} \tag{14}$$

Thus, from *Eq. (14)*, we have

$$\|L_3\| \le CN^{-1}\|\tilde{G}\|. \tag{15}$$

To find over bound for $\|L_4\|$, similar to process of obtaining *Eq. (15)*, we have

$$\|L_4\| \le CN^{-1} \left\| \tilde{G}\left(\eta, I_N^C\big(\tilde{z}(\eta)\big)\right) - \tilde{G}\big(\eta, \tilde{z}(\eta)\big) \right\|. \tag{16}$$

Since $\tilde{G}\big(\eta, \tilde{z}(\eta)\big)$ satisfies in the Lipschitz condition with respect to the second variable. Thereby from *Eq. (16)*, we have

$$\|L_4\| \le CN^{-1}L\|e\|. \tag{17}$$

Where L is the Lipschitz constant. Now, applying (5.5.22) from [29] for *Eq. (17)*, we have

$$\|L_4\| \le CN^{-1-m}\, L\, |\tilde{z}|_{H_w^{m;N}(-1,1)}.$$

Using Lipschitz condition and (5.5.22) from [29], we have

$$\| \int_{-1}^{\zeta} \widetilde{R}(\zeta,\eta) \left(\widetilde{G}\left(\eta, I_N^C(\tilde{z}(\eta))\right) - \widetilde{G}\left(\eta, \tilde{z}(\eta)\right)\right) d\eta \| \leq CLN^{-m} |\tilde{z}|_{H_w^{m;N}(-1,1)}.$$

Combining the above estimates with *Eq. (11)* leads to the intended error estimate *Eq. (6)*.

4 | Numerical Results

In this section, we use the proposed method for solving some numerical examples to test the accuracy of the method. We get the numerical results by Wolfram Mathematica 12.2. The presented method has been compared with the Legendre Tau method in [27]. For simplicity, we use the notation $\|.\|$ instead of $\|.\|_{L_w^2}$.

Example 1. Consider the following pantograph-type Volterra Hammerstein integral equation [27]

$$z(s) = g(s) + \int_0^{qs} R(t,s) G\left(t, z(t)\right) dt, \qquad s \in [0,1].$$

Where

$$\begin{cases} g(s) = -\dfrac{1}{4}\exp(s)(s - \sin(s)) + \tan(s), \\ q = \dfrac{1}{2}, \dfrac{7}{10}, \\ R(s,t) = \exp(s)\cos^2(t), \\ G\left(t, z(t)\right) = z^2(t). \end{cases}$$

And the exact solution is $z(s) = tan(s)$. The function $G(t, z(t))$ satisfies in the Lipschitz condition with respect to the second variable.

Table 1 contains a comparison of the L^2-norms of the error function between the presented method and the Legendre Tau method in [27] for *Example 1* with $q = \frac{1}{2}$. *Figs. 1* and *2* show the plots of the obtained errors $|z - I_N^C(z)|$ by the presented method in *Example 1* for $N = 8$ and $N = 32$, respectively, with $q = \frac{1}{2}$. *Fig. 9* shows tending $\|z - I_N^C(z)\|$ to zero by increasing N in *Example 1* with $q = \frac{1}{2}$.

Table 2 contains the obtained errors $\|z - I_N^C(z)\|$ by the presented method for *Example 1* with $q = \frac{7}{10}$. *Figs. 3* and *4* show the plots of the obtained errors $|z - I_N^C(z)|$ by the presented method in *Example 1* for $N = 8$ and $N = 32$, respectively, with $q = \frac{7}{10}$. *Fig. 10* shows tending $\|z - I_N^C(z)\|$ to zero by increasing N in *Example 1* with $q = \frac{7}{10}$.

Example 2. Consider the following pantograph-type Volterra Hammerstein integral equation [27]

$$z(s) = g(s) + \int_0^{qs} R(t,s) G\left(t, z(t)\right) dt, \qquad s \in [0,1].$$

Where

$$\begin{cases} g(s) = \dfrac{1}{8}\left(4\cosh\left(\dfrac{3s}{4}\right) - 4\cosh(s) + s\sinh\left(\dfrac{3s}{4}\right)\right) + \sinh^{-1}(\frac{s}{2}), \\ q = \dfrac{1}{4}, \\ R(s,t) = \cosh(s-t), \\ G\left(t, z(t)\right) = \sinh\left(z(t)\right). \end{cases}$$

And the exact solution is $z(s) = sinh^{-1}(\frac{s}{2})$. The function $G(t, z(t))$ satisfies in the Lipschitz condition with respect to the second variable.

Table 3 contains a comparison of the L^2-norms of the error function between the presented method and the Legendre Tau method in [27] for *Example 2*. *Figs. 5* and *6* show the plots of the obtained errors $|z - I_N^C(z)|$ by the presented method for *Example 2* with $N = 8$ and $N = 32$, respectively. *Fig. 11* shows tending $\|z - I_N^C(z)\|$ to zero by increasing N in *Example 2*.

Example 3. Consider the following pantograph-type Volterra Hammerstein integral equation

$$z(s) = g(s) + \int_0^{qs} R(t,s)G\big(t, z(t)\big)dt, \qquad s \in [0,1].$$

Where

$$\begin{cases} g(s) = 2.125 - 0.42s^2 + \cos\left(\frac{s}{4}\right) - 2\cos\left(\frac{2s}{5}\right) - 0.125\cos\left(\frac{4s}{5}\right) - 1.8s\sin\left(\frac{2s}{5}\right) - \\ 0.225s\sin\left(\frac{4s}{5}\right), \\ q = \frac{8}{10}, \\ R(s,t) = s + t, \\ G\big(t, z(t)\big) = z(t)^4. \end{cases}$$

And the exact solution is $z(s) = cos\left(\frac{s}{4}\right)$. The function $G(t, z(t))$ satisfies in the Lipschitz condition with respect to the second variable.

Table 4 contains the obtained errors $\|z - I_N^C(z)\|$ by the presented method for *Example 3*. *Figs. 7* and *8* show the plots of the obtained errors $|z - I_N^C(z)|$ by the presented method in *Example 3* for $N = 8$ and $N = 32$, respectively. *Fig. 12* shows tending $\|z - I_N^C(z)\|$ to zero by increasing N in *Example 3*.

Table 1. Comparison of the presented method and the Legendre Tau method in [27] for example 1 with $q = \frac{1}{2}$.

N	$\|z - I_N^C(z)\|$ for the Presented Method	$\|z - I_N^C(z)\|$ for the Legendre Tau Method
2	4.99e -02	−−
4	1.97e -03	6.17e − 02
8	1.19e −05	7.96e − 03
16	2.03e -10	1.62e − 04
32	1.29e −15	8.66e − 08
64	8.66e -16	3.31e − 14

Table 2. The obtained errors $\|z - I_N^C(z)\|$ by the presented method for example 1 with $q = \frac{7}{10}$.

N	2	4	8	16	32	64
$\|z - I_N^C(z)\|$	6.01e -02	1.50e − 03	1.23e − 05	1.97e − 10	8.77e − 16	1.74e − 16

Table 3. Comparison of the presented method and the Legendre Tau method in [27] for example 2.

N	$\|z - I_N^C(z)\|$ for the Presented Method	$\|z - I_N^C(z)\|$ for the Legendre Tau Method
2	4.23e -13	−−
4	1.69e −14	6.29e − 04
8	8.49e -16	1.17e − 05
16	5.69e -16	1.24e − 08
32	3.95e -16	4.94e − 14
64	1.04e −16	−−

Table 4. The obtained errors $\|z - I_N^C(z)\|$ by the presented method for example 3.

N	2	4	8	16	32	64
$\|z - I_N^C(z)\|$	2.85e -04	1.29e -07	1.35e −13	8.30e -14	1.62e −15	1.01e −15

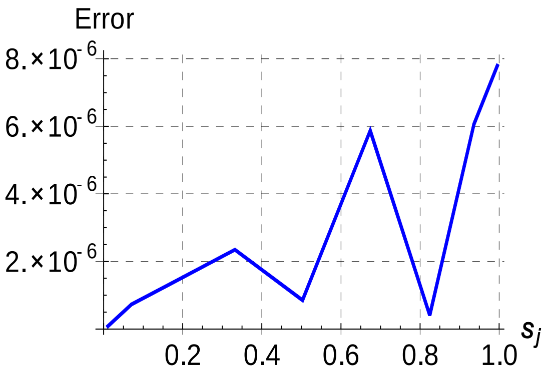

Fig. 1. Plot of the obtained errors $|z - I_N^C(z)|$ by the presented method for N = 8 in example 1 with $q = \frac{1}{2}$.

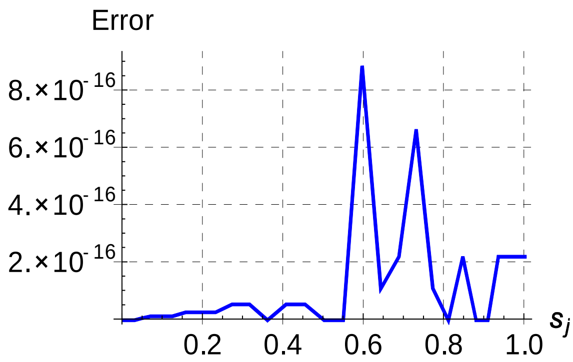

Fig. 2. Plot of the obtained errors $|z - I_N^C(z)|$ by the presented method for N = 32 in example 1 with $q = \frac{1}{2}$.

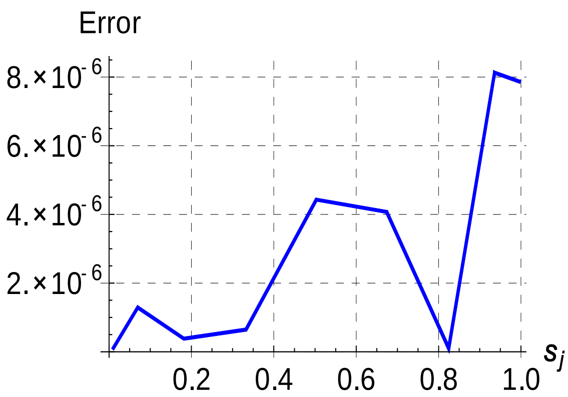

Fig. 3. Plot of the obtained errors $|z - I_N^C(z)|$ by the presented method for N = 8 in example 1 with $q = \frac{7}{10}$.

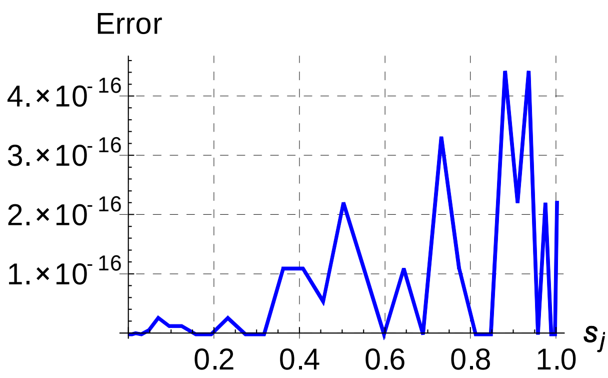

Fig. 4. Plot of the obtained errors $|z - I_N^C(z)|$ by the presented method for $N = 32$ in example 1 with $q = \frac{7}{10}$.

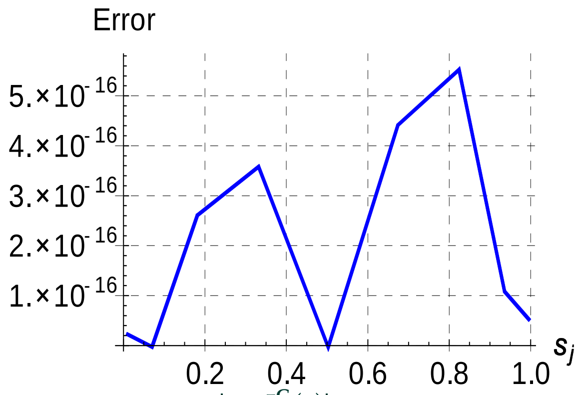

Fig. 5. Plot of the obtained errors $|z - I_N^C(z)|$ by the presented method for $N = 8$ for example 2.

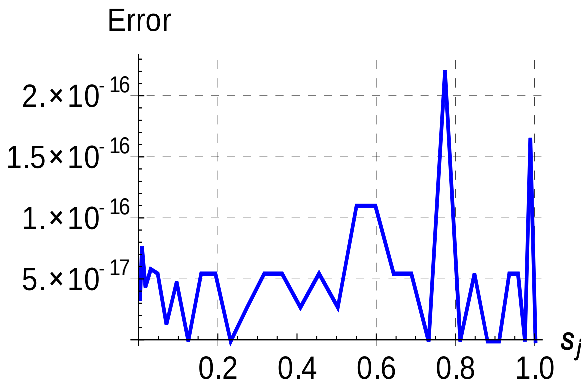

Fig. 6. Plot of the obtained errors $|z - I_N^C(z)|$ by the presented method for $N = 32$ for example 2.

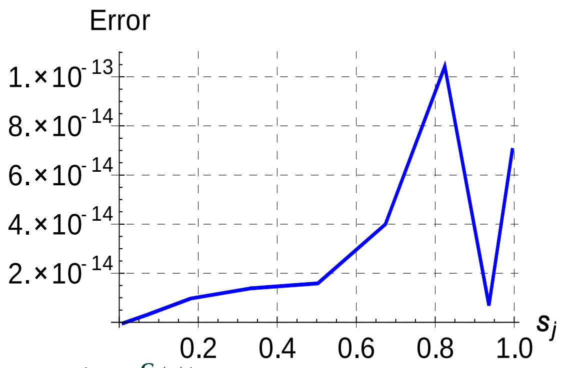

Fig. 7. Plot of the obtained errors $|z - I_N^C(z)|$ by the presented method for $N = 8$ for example 3.

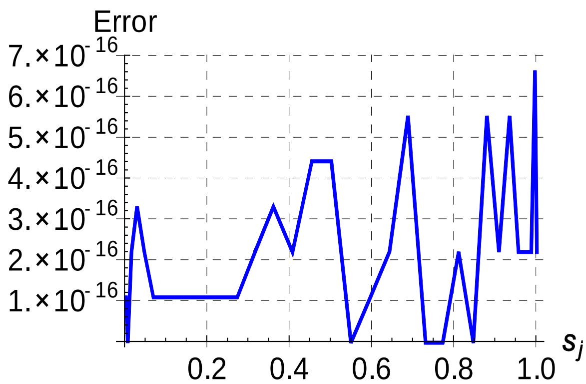

Fig. 8. Plot of the obtained errors $|z - I_N^C(z)|$ by the presented method for $N = 32$ for example 3.

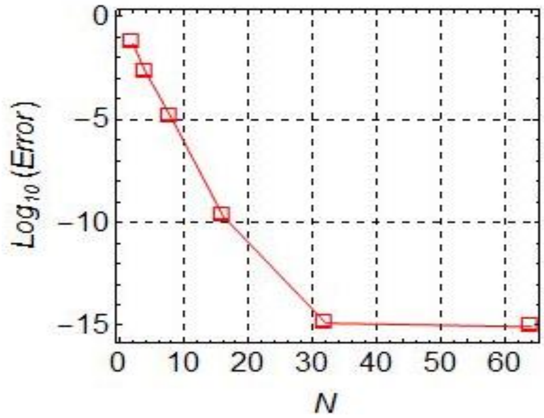

Fig. 9. Graph of tending $\|z - I_N^C(z)\|$ to zero by increasing N in example 1 with $q = \frac{1}{2}$.

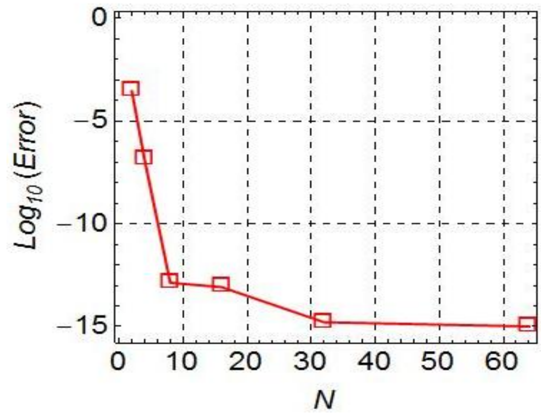

Fig. 10. Graph of tending $\|z - I_N^C(z)\|$ to zero by increasing N in example 1 with $q = \frac{7}{10}$.

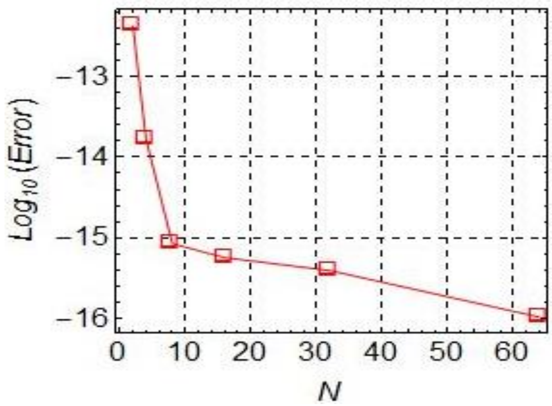

Fig. 11. Graph of tending $\|z - I_N^C(z)\|$ to zero by increasing N in example 2.

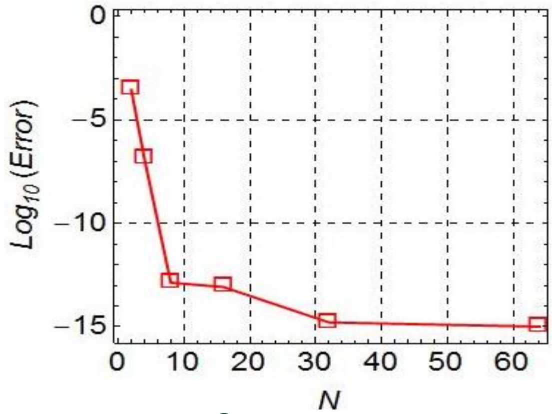

Fig. 12. Graph of tending $\|z - I_N^C(z)\|$ to zero by increasing N in example 3.

5 | Conclusion

In this work, we applied a spectral collocation method to the numerical solution of the pantograph-type Volterra Hammerstein integral equations based on the first kind Chebyshev polynomials. We employed the Lagrange interpolating polynomial to approximate the solution.

The convergence of the presented method is analyzed. Some numerical examples are prepared to test the accuracy of the proposed method. We observed that the proposed method is more accurate than Legendre Tau method in [27]. The other polynomials can be used for solving these equations such as Chelyshkov polynomials, Legendre polynomials, other kinds of Chebyshev polynomials, and so on.

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Conflicts of Interest

The authors have no conflict of interest.

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