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Trigonometric Functions and Some Trigonometrical Identities

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Abstract

Trigonometry is a part of mathematics which deals with specific functions of angles and their application, and it is studies of the dependence between the sides and angles of a triangle. The word trigonometry is associated the operations between the sides and angles of the triangles. Initially, it aimed to calculate the values of all elements of a triangle (height, centroid length, bisector, radius, area and angles) using data sufficient to determine the triangle. In the paper, trigonometric functions are defined, and some statements about trigonometric identities are stated and proved.

Keywords: Mathematics, Trigonometric functions, Trigonometric identities.

1 | Introduction

Trigonometry is a mathematical discipline that deals with trigonometric functions and their applications. The roots of this mathematical discipline can be found even in the old century in Egypt and Babylon. The word trigonometry itself is composed of the Greek words trigonon (triangle) and meron (measure), which indicates that this discipline initially dealt with the problem of measuring a triangle, i.e. its sides and angles. Modern trigonometry deals with the trigonometric functions of angles and numbers between which there is a close connection. To understand this work, knowledge of the basic properties of trigonometric functions is sufficient. It is also important to know the difference between an identity and an equation. An identity is an equation that is true for any values of the unknowns. Equations in the general case cannot be added if we want to save earlier solutions, while identities can, let's look at it with an obvious example. We recite some research papers and books [1]-[13] for the right direction from trigonometric functions and its identities.

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The solutions of the first equation are $x^1=1$ and $x^2=2$. The solution to the second equation is $x^1=4$. When we add the first two equations, we get a third one, and its solutions are obviously not 1, 2, or 4. The sum of finitely many identities is an identity. An identity multiplied by any number remains an identity. I will derive all the following formulas using the aforementioned properties of identities, so that all starting and concluding formulas are identities. Numerous trigonometric identities are difficult to learn by heart, which is unfortunately the practice in most high schools, so among students, even those who are better at mathematics and want to enter technical colleges, trigonometry is considered the "hardest" field. I believe that attention is mostly focused on solving specific tasks and that the theoretical background of identities and why they are the way they are, as well as the whole essence and purpose of trigonometry, is neglected.

2 | Applications of Trigonometry

Those who say that trigonometry is not needed in real life are not needed on the road. So, what are its common applied tasks? measure the distance between inaccessible objects.

The triangle technique is of great importance, which enables the measurement of distances to nearby stars in astronomy, between benchmarks in geography, control satellite navigation systems. The use of trigonometric techniques should also be noted, such as navigation techniques, music theory, acoustics, optics, financial market analysis, electronics, probability theory, statistics, biology, medicine (ultrasound and computed tomography), pharmacology, chemistry, number theory (and, as a result, cryptology, seismology, meteorology, oceanology, cartography, many parts of physics, topography and geodesy, architecture, phonetics, economics, electronic equipment, mechanical engineering, computer graphics, crystallography, etc.

3 | Trigonometrical Functions

In mathematics, trigonometric functions (also called circular functions, angle functions, or trigonometric functions) are real functions that relate the angle of a right triangle to the ratio of two side lengths. They are widely used in all sciences related to geometry, such as navigation, mechanics of solid bodies, celestial mechanics, geodesy and many others. They are among the simplest periodic functions and as such are also widely used to study periodic phenomena through Fourier analysis.

The most commonly used trigonometric functions in modern mathematics are sine, cosine and tangent. Their reciprocals are cosecant, secant, and cotangent, which are less commonly used. Each of these six trigonometric functions has a corresponding inverse function and an analog among the hyperbolic functions.

The oldest definitions of trigonometric functions, which refer to right triangles, define them only for acute angles. To extend the sine and cosine functions to functions whose domain is the entire real line, geometric definitions using the standard unit circle (i.e., a circle of radius 1 unit) are often used; then the domain of the other functions is a straight line with some isolated points removed. Modern definitions express trigonometric functions as infinite series or as solutions of differential equations. This allows extending the domain of sine and cosine functions to the entire complex plane, and the domain of other trigonometric functions to the complex plane with some isolated points removed.

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, radians are a more natural measurement because they are related directly to the unit circle, a circle with radius 1.

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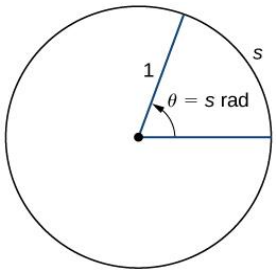

Fig. 1. The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

The radian measure of an angle is defined as follows. Given an angle θ, let s be the length of the corresponding arc on the unit circle (*Fig. 1*). We say the angle corresponding to the arc of length 1 has radian measure 1.

4 | Trigonometrical Identities

In mathematics, the trigonometric identities are equivalent to the use of trigonometric functions and they hold for every value of the variables. Geometrically, they are identities involving certain functions of one or more angles. They are special trigonometric identities, they include both angles of length of the sides of the triangle. Only some are mentioned in this paper.

These identities are useful whenever we have an expression involving trigonometric functions that needs to be simplified. An essential requirement is the integration of non-trigonometric functions. A common technique involves first applying the substitution rule to trigonometric functions, and then simplifying the integral results with trigonometric identities.

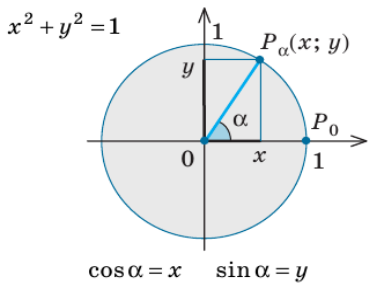

Fig. 2. A unit circle with radius 1 centered at the origin.

Basic trigonometric identity $\sin^2 \alpha + \cos^2 \alpha = 1$,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \quad 1 + \tan^2\alpha = \frac{1}{\cos^2\alpha},$$
$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}, \quad 1 + \cot^2\alpha = \frac{1}{\sin^2\alpha},$$
$$\tan\alpha \,.\cot\alpha = 1.$$

The picture at the beginning of the lecture shows the unit circle, i.e. a circle of radius 1 with the center at the beginning. The equation for this circuit is: $x^2 + y^2 = 1$.

Some α point P_0 (1;0) unit circle transitions to a point as a consequence of rotation by P α(x;y) angle (that is, as a consequence of the rotation of the angle, α the radius OP_0 turns into a radius OP α). Recall that the sine is the ordinate α of a point P α(x;y) unit circles, that is $\sin \alpha = y$, and cosine α is the abscissa of this point, that is $\alpha = x$. The coordinates P α of the point satisfy the equation of the circle, then $x^2 + y^2 = 1$. So, $\sin^2 \alpha + \cos^2 \alpha = 1$. This relation is called by the basic trigonometric identity.

Let's also remember that:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad (\cos\alpha \neq 0)\,;\ \cot\alpha = \frac{\cos\alpha}{\sin\alpha}, \qquad (\sin\alpha \neq 0).$$

Then

$$\tan\alpha\,.\cot\alpha = \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\cos\alpha}{\sin\alpha} = 1.$$

Using these relations and the basic trigonometric identity,

$$1+\tan^2\alpha = 1+\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha},$$

$$1+\tan^2\alpha = \frac{1}{\cos^2\alpha},\ \cos\alpha \neq 0).$$

Similarly,

$$1+\cot^2\alpha = 1+\frac{\cos^2\alpha}{\sin^2\alpha} = \frac{\cos^2\alpha+\sin^2\alpha}{\sin^2\alpha} = \frac{1}{\sin^2\alpha},$$

$$1+\cot^2\alpha = \frac{1}{\sin^2\alpha},\ \sin\alpha \neq 0).$$

Example 1. Knowing the value of one trigonometric function and the interval in which it is located α, find the values of the remaining three trigonometric functions:

I. $\sin\alpha = \frac{4}{5},\ 90^o < \alpha < 180^o.$

II. $\tan\alpha = \frac{1}{3},\ \pi < \alpha < \frac{3\pi}{2}.$

Comment 1. Equality $\sin^2\alpha + \cos^2\alpha = 1$ connects $\sin\alpha$ and $\cos\alpha$ and allows one of these functions to be expressed in terms of the other.

For example, $\cos^2\alpha = 1\text{-}\sin^2\alpha$. Then $\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$. Given the quarter in which α, we can determine the sign that must be taken on the right side of the formula (this is the sign of the cosine in the second quarter). Knowing $\sin\alpha$ and $\cos\alpha$, we find $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$. Note that after finding $\tan\alpha$, the value $\cot\alpha$ can also be found from the relation $\tan\alpha\cdot\cot\alpha = 1$.

Comment 2. Equality $\tan\alpha\cdot\cot\alpha = 1$ connects $\tan\alpha$ and $\cot\alpha$ allows one of these functions to be expressed in terms of the other as a reciprocal.

Equality $1+\tan^2\alpha = \frac{1}{\cos^2\alpha}$ connects $\tan\alpha$ and $\cos\alpha$ allows one of these functions to be expressed in terms of the other.

For example, $\cos^2\alpha = \frac{1}{1+\tan^2\alpha}$. Then $\cos\alpha = \sqrt{\frac{1}{1+\tan^2\alpha}}$.

Knowing in which quarter it is α, we can determine the sign that must be taken on the right side of the formula (this is the sign of the cosine in the third quarter). To find $\sin\alpha$, you can use relationship:

$$\tan\alpha\,.\cos\alpha = \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha = \sin\alpha.$$

The solution:

I. From equality $\sin^2\alpha + \cos^2\alpha = 1$, we get $\cos^2\alpha = 1 - \sin^2\alpha$, from here:

$$\cos^2\alpha = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}.$$

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Since $90° < \alpha < 180°$, that $\cos\alpha < 0$ and $\cos\alpha = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$. Then

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}, \quad \cot\alpha = \frac{\cos\alpha}{\sin\alpha} = -\frac{3}{4}.$$

II. From equality $\tan\alpha \cdot \cot\alpha = 1$, we get $\cot\alpha = \frac{1}{\tan\alpha} = 3$ and $1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}$.

Substitutes value into equality $\tan\alpha$ and gets $1 + \frac{1}{9} = \frac{1}{\cos^2\alpha}$. From here $\cos^2\alpha = \frac{9}{10}$.

Since $\pi < \alpha < \frac{3\pi}{2}$, *then* $\cos\alpha = -\sqrt{\frac{9}{10}} = -\frac{3}{\sqrt{10}} < 0$, therefore

$$\sin\alpha = \tan\alpha \,.\, \cos\alpha = \frac{1}{3}\left(-\frac{3}{\sqrt{10}}\right) = -\frac{1}{\sqrt{10}} < 0.$$

Example 2. Shorten the expression $\sin^4\alpha - \cos^4\alpha$.

Comment 3. For the transformation of trigonometric expressions, in addition to trigonometric formulas, algebraic formulas are also used, especially abbreviated formulas for multiplication. So, the expression $\sin^4 a - \cos^4 a$ can be viewed as a difference of squares $(\sin^2 a)^2 - (\cos^2 a)^2$. Then it can be decomposed into factors (into the product of sum and difference $\sin^2 a$ and $\cos^2 a$).

Solution:

$$\sin^4\alpha - \cos^4\alpha + \cos^2\alpha = (\sin^2\alpha - \cos^2\alpha)(\sin^2\alpha + \cos^2\alpha) + \cos^2\alpha =$$
$$(\sin^2\alpha - \cos^2\alpha), 1 + \cos^2\alpha = \sin^2\alpha - \cos^2\alpha + \cos^2\alpha = \sin^2\alpha.$$

5 | Conclusion

In this paper you will encounter complex trigonometric expressions. Often, complex trigonometric expressions can be equivalent to less complex expressions. The process of showing that two trigonometric expressions are equivalent (regardless of the value of the angle) is called proving trigonometric identities.

There are several lines of reasoning that we use when proving the trigonometric identity. Often, one of the steps in proving the identity is to change each expression to sine and cosine.

Example 3. Prove the identity: $\operatorname{cosec} x \cdot \tan x = \sec x$.

Solution: we will arrange the left side.

$$\operatorname{cosec} x \ \tan x = \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} = \frac{1}{\cos x} = \sec x.$$

We ended up with the same thing as on the right, so we know the identity is correct.

When working with identities, unlike equations, substitutions and mathematical operations are performed on only one side of the identity. The thought process for determining identity goes in the direction of looking at both sides of equality, and the goal is to show that both sides can be transformed into the other.

Use all known trigonometric identities available.

$$\operatorname{cosec} x \ \tan x = \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} = \frac{1}{\cos x} = \sec x.$$

Conflicts of Interest

All co-authors have seen and agree with the contents of the manuscript and there are no conflicts of interest to report. We certify that the submission is original work and is not under review at any other publication.

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