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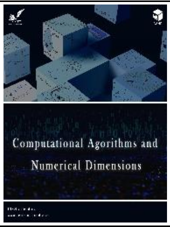


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# Enhanced a Novel Approach for Smoothing Data in Modeling and Decision-Making Problems under Fuzziness

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## Abstract

Smoothing (filtering) of data is a major problem in engineering and science. In this paper, the smoothing of data arising in modelling and decision-making problems is considered. Firstly, the conventional smoothing and filtering problem and its extension to a fuzzy situation are introduced.

**Keywords:** Smoothing problem, Piecewise quadratic fuzzy numbers, Close interval approximation, Probability function, Pascal law, Geometrical modified law, Forecasting.

## 1 | Introduction



Optimization applications abound in many areas of science and engineering [1], [2]. In real practice, some parameters involved in optimization problems are subject to uncertainty for various reasons, including estimation errors and unexpected disturbance [3]. Such uncertain parameters can be product demands in process planning [4], kinetic constants in reaction separation-recycling system design [5] and task durations in batch process scheduling [6], among others. The issue of uncertainty could, unfortunately, render the solution of a deterministic optimization problem (i.e., the one disregarding uncertainty) suboptimal or even infeasible [7]. The infeasibility, i.e., the violation of constraints in optimization problems, has a disastrous consequence on the solution quality. Motivated by practical concern, optimization under uncertainty has attracted tremendous attention from academia and industry [3], [8].

The Smoothing problem (not to be confused with smoothing in statistics, image processing and other contexts) refers to Recursive Bayesian estimation, also known as Bayes filter, which is the problem of estimating an unknown probability density function recursively over time using incremental incoming measurements. It is one of the main problems defined by Norbert Wiener (Note: Do not be confused with blurring and smoothing using methods such as moving average). The stochastic

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filtering problem has caught the attention of thousands of mathematicians, engineers, statisticians, and computer scientists. Its applications span the whole spectrum of human endeavour, including satellite tracking, credit risk estimation, human genome analysis, and speech recognition. Stochastic filtering has engendered a surprising number of mathematical techniques for its treatment and has played an essential role in developing new research areas, including stochastic partial differential equations, stochastic geometry, rough paths theory, and Maldivian calculus. It also spearheaded research in classical mathematics, such as Lie algebras, control theory, and information theory. This paper aims to give a brief historical account of the subject, concentrating on the continuous-time framework. Enormous authors studied smoothing problem (for instance, Cosme et al. [9], Einicke [10], Niu et al. [11] and Browarka et al. [12]).

One of the difficulties that occur in the application of mathematical programming is that the parameters in the formulation are not constants but uncertain. Fuzzy set theory was first introduced by Zadeh [13]. Fuzzy numerical data can be represented by employing fuzzy subsets of the real line known as fuzzy numbers. Dubois and Prade [14] have extended the use of the algebraic operations on real numbers to fuzzy numbers using a fuzzification principle. The fuzzy nature of a goal programming problem was discussed by Zimmermann [15].

Badawi et al. [16] developed a smoothing theory for finite-dimensional linear stochastic systems in the context of stochastic realization theory. The basic idea is to embed the given stochastic system in a class of similar systems with the same output process and Kalman-Bucy filter. Klibanoff et al. [17] proposed and characterized a smooth model of decision-makers under ambiguity. Valenzuela and Pasadas [18] developed a new methodology for defining error and similarity measure indexes to establish an adequate criterion for comparison function approximation using fuzzy data. Wan and Hu [19] introduced a method to optimize the fused track quality in the intelligence network of radar target fusion systems. Ning and You [20] proposed a novel, data-driven, robust optimization framework that leverages the power of machine learning and big data analytics for uncertainty-free decision-making.

Nowadays, a wide array of emerging machine-learning tools can be leveraged to analyze data and extract accurate, relevant, and helpful information to facilitate knowledge discovery and decision-making. Deep learning, one of the most rapidly growing machine learning subfields, demonstrates remarkable power in deciphering multiple layers of representations from raw data without any domain expertise in designing feature extractors [21]. More recently, the dramatic progress of mathematical programming [22], coupled with recent advances in machine learning [23], especially in deep learning over the past decade [24] sparked a flurry of interest in data-driven optimization [25]–[27].

The rest of the paper is outlined as follows.

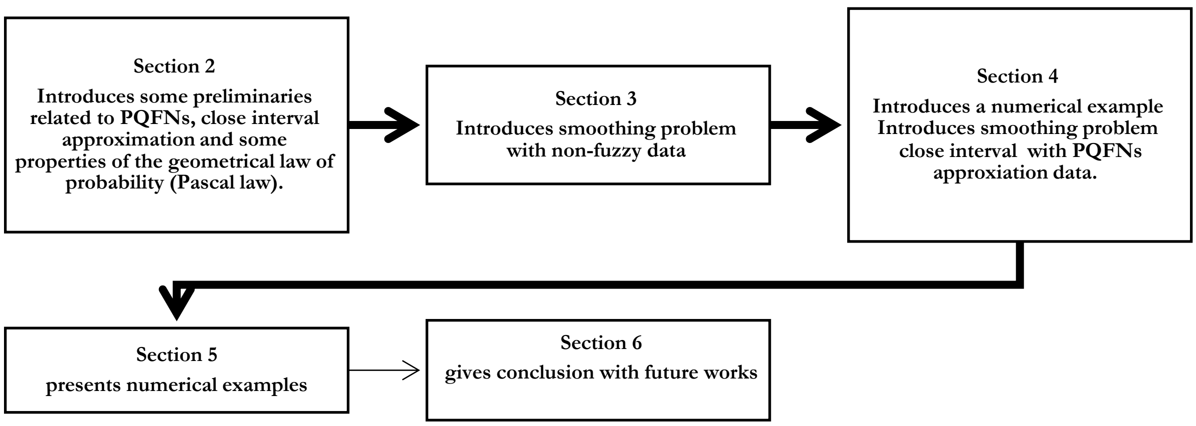


**Fig. 1. Layout of remaining paper.**

## 2 | Preliminaries

To discuss the problem easily, it recalls basic rules and findings related to fuzzy numbers, Piecewise Quadratic Fuzzy Numbers (PQFNs), close interval approximation, and its arithmetic operations.

**Definition 1 ([13]).** A fuzzy number $A$ is a fuzzy set with a membership function defined as $\pi_A\ x): \Re \rightarrow [0,1]$, and satisfies:

I. $A$ is fuzzy convex, i.e., $\pi_A\ \delta\, x + \ 1-\delta)\, y) \geq min\{\pi_A\ x), \pi_A\ y)\}$; $for\ all\ x, y \in\ \Re\ ; 0 \leq \delta \leq 1$.

II. $A$ is normal, i.e., $\exists\ x_0 \in \Re$ for which $\pi_A\ x_0) = 1$.

III. $Supp\left(A\right) = \{x \in \Re: \pi_A\ x) > 0\}$ is the support of $\widetilde{A}$.

IV. $\pi_A(x)$ is an upper semi-continuous (i.e., for each $\alpha \in\ 0{,}1)$, the $\alpha -$ cut set $A_\alpha = \{x \in \Re: \pi_A \geq \alpha\}$ is closed.

**Definition 2 ([28]).** A PQFN is denoted by $A_{PQ} =\ a_1, a_2, a_3, a_4, a_5)$, where $a_1 \leq\ a_2 \leq a_3 \leq\ a_4 \leq\ a_5$ are real numbers and is defined by if its membership function $\mu_{A_{PQ}}$ is given by (see *Fig. 1*)

$$\mu_{\widetilde{A}_{PQ}} = \begin{cases} 0, & x < a_1, \\ \frac{1}{2}\frac{1}{a_2 - a_1)^2}\ x - a_1)^2, & a_1 \leq x \leq a_2, \\ \frac{1}{2}\frac{1}{a_3 - a_2)^2}\ x - a_3)^2 + 1, & a_2 \leq x \leq a_3, \\ \frac{1}{2}\frac{1}{a_4 - a_3)^2}\ x - a_3)^2 + 1, & a_3 \leq x \leq a_4, \\ \frac{1}{2}\frac{1}{a_5 - a_4)^2}\ x - a_5)^2, & a_4 \leq x \leq a_5, \\ 0, & x > a_5. \end{cases}$$

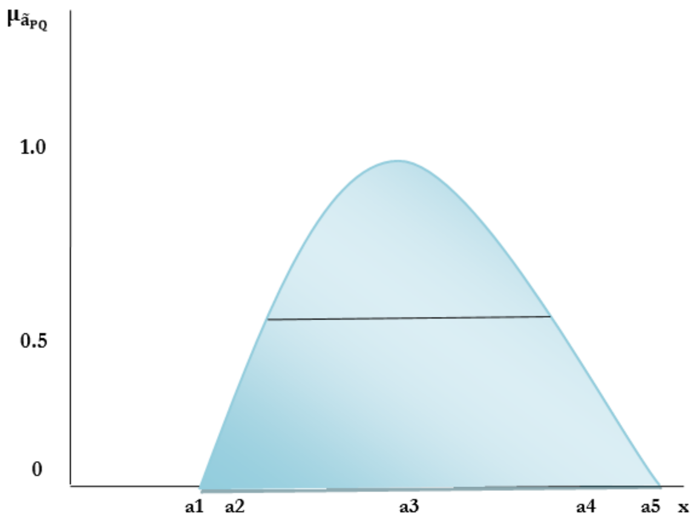


**Fig. 2. Graphical representation of a PQFN.**

The interval of confidence at level $\alpha$ for the PQFN is defined as

$$\left(\widetilde{A}_{PQ}\right)_\alpha = [a_1 + 2\ a_2 - a_1)\alpha, a_5 - 2\ a_5 - a_4)\alpha]; \text{for all } \alpha \in [0,1].$$

**Definition 3 ([28]).** Let $A_{PQ} =\ a_1, a_2, a_3, a_4, a_5)$ and $B_{PQ} =\ b_1, b_2, b_3, b_4, b_5)$ be two PQFNs. The arithmetic operations on $A_{PQ}$ and $B_{PQ}$ are:

I. Addition: $A_{PQ}(+)B_{PQ} = \ a_1 + b_1, a_2 + \ b_2, a_3 + \ b_3, a_4 + b_4, a_5 + b_5)$.

II. Subtraction: $A_{PQ} \ -)B_{PQ} = \ a_1 + b_5, a_2 + \ b_4, a_3 + \ b_3, a_4 + b_2, a_5 + b_1)$.

III. Scalar multiplication: $kA_{PQ} = \begin{cases} \ k\,a_1, k\,a_2, k\,a_3, k\,a_4, k\,a_5), k > 0, \\ \ k\,a_5, k\,a_4, k\,a_3, k\,a_2, k\,a_1), k < 0. \end{cases}$

**Definition 4 ([28]).** An interval approximation $[A] = \left[a_\alpha^L, a_\alpha^U\right]$ of a PQFN $A$ is called closed interval approximation if $a_\alpha^L = inf\left\{x \in \mathbb{R}\text{: } \mu_A \geq 0.5\right\}$, *and* $a_\alpha^U = sup\left\{x \in \mathbb{R}\text{: } \mu_A \geq 0.5\right\}$.

**Definition 5 ([28]). Associated ordinary numbers.** If $[A] = [a_\alpha^-, a_\alpha^+]$ is the close interval approximation of PQFN, the Associated ordinary number of $[A]$ is defined as $A = \frac{a_\alpha^- + a_\alpha^+}{2}$, or $A = \frac{a_1 + a_2 + 2a_3 + a_4 + a_5}{6}$.

**Definition 6 ([28]).** Let $[A] = \left[a_\alpha^L, a_\alpha^U\right]$, and $[B] = \left[b_\alpha^L, b_\alpha^U\right]$ be two interval approximations of PQFN, then the algebraic operations are:

I. Addition: $[A](+)[B] = \left[a_\alpha^L + b_\alpha^L, a_\alpha^U + b_\alpha^U\right]$.

II. Subtraction: $[A] \ -)[B] = \left[a_\alpha^L - b_\alpha^U, a_\alpha^U - b_\alpha^L\right]$.

III. Scalar multiplication: $\alpha\ [A] = \begin{cases} \left[\alpha\, a_\alpha^L,\ \alpha\, a_\alpha^U\right], \alpha > 0, \\ \left[\ \alpha\, a_\alpha^U, \alpha\, a_\alpha^L\right], \alpha < 0. \end{cases}$

IV. Multiplication: $[A](\times)[B] = \left[\frac{a_\alpha^U b_\alpha^L + a_\alpha^L\, b_\alpha^U}{2}, \frac{a_\alpha^L\, b_\alpha^L + a_\alpha^U\, b_\alpha^U}{2}\ \right]$.

V. Division: $[A]\ \div)[B] = \begin{cases} \left[2\left(\frac{a_\alpha^L}{b_\alpha^L + b_\alpha^U}\right), 2\left(\frac{a_\alpha^U}{b_\alpha^L + b_\alpha^U}\right)\right], [B] > 0, b_\alpha^L + b_\alpha^U \neq 0, \\ \left[2\left(\frac{a_\alpha^U}{b_\alpha^L + b_\alpha^U}\right), 2\left(\frac{a_\alpha^L}{b_\alpha^L + b_\alpha^U}\right)\right], [B] < 0, b_\alpha^L + b_\alpha^U \neq 0. \end{cases}$

**Definition 7.** The probability function defines the geometrical law of probability (Pascal law).

$$\Pr\ v) = \gamma\ 1 - \gamma)^v, 0 < \gamma < 1, v \in \mathbb{N}, \tag{1}$$

with Mean $\mu$ and Variance $\sigma_v^2$ are given by

$$\mu = E\ v) = \frac{1 - \delta}{\delta}, \tag{2}$$

$$\sigma_v^2 = \frac{1 - \gamma)^2}{\gamma^2}, \tag{3}$$

where $v$ is the random variable having values $v \in \mathbb{N}$.

**Definition 8.** A less classical law (or geometrical modified law) truncated to the left of $v = 0$ and is defined by

$$\Pr\ v) = \begin{cases} \ 1 - \gamma)^s, & v = 0, \\ \gamma\ 1 - \gamma)^{s-v}, & 0 < v < s, \\ 0, & v > s, v, s \in \mathbb{N}. \end{cases} \tag{4}$$

## 3 | Smoothing with Non-Fuzzy Data

Based on the *Probability Law (4),* the exponential smoothing method is established, which deals with the effect of the numerical value on a chronological (time) sequence and with some coefficients corresponding to the probability law. The estimate of $y(t)$ at the time $t$ is given by the following convolution summation as

$$y\ t) = \ 1-\gamma)^{t}.x\ 0) + \gamma\ 1-\gamma)^{t-1}.x\ 1) + \gamma\ 1-\gamma)^{t-2}.x\ 2) + \cdots + \gamma\ 1-\gamma).x\ t-1) +$$

$$\gamma\ x(t)= 1-\gamma)^{t}.x\ 0) + \textstyle\sum_{v=1}^{t}\gamma\ 1-\gamma)^{t-v}.x\ v) = \ 1-\gamma)^{t+1}x\ 0) + \sum_{v=1}^{t}\gamma\ 1-\gamma)^{t-v}.x\ v). \tag{5}$$

By using the *Eq. (5)*, $1-\gamma)y\ t-1)$ can be computed, and a recursive formula is given as

$$y\ v+1) = \ 1-\gamma)y\ v) + \gamma\ x\ v+1), 0 \leq v \leq t-1, y\ 0) = x\ 0), \tag{6}$$

Which can be used for the computation of the estimates. The convolution summation process of *Eq. (5)* is, in a way, a "discounting" of the past data, which geometrically decreases the values of the chronological sequence from $t$ to 0 with the accumulation of the remaining weights starting at $t = 0$.

The estimation of $y\ v+1)$ is determined from $x\ v+1)$ and $y(v)$ as given by *Eq. (6)*. Often, we are interested in the extrapolation $\gamma\ t+1)$ at the time $t+1)$ which can be obtained from $y(t)$ using a special hypothesis.

There are several examples of various orders of smoothing:

I. Single smoothing of order 1: Using *Eq. (5)*, this is given by

$$y^{\ 1)}\ t) = \ 1-\gamma)^{t+1}.x\ 0) + \sum_{v=0}^{t}\gamma\ 1-\gamma)^{t-v}.x\ v). \tag{7}$$

II. Over-smoothing of order 2

$$y^{\ 2)}\ t) = \ 1-\gamma)^{t+1}.y^{\ 1)}\ 0) + \sum_{v=1}^{t}\gamma\ 1-\gamma)^{t-v}.y^{\ 1)}\ v), \tag{8}$$

where $y^{\ 1)}\ t)$, is given in *Eq. (7)*.

This over-smoothing can also be expressed in terms of $x\ v)$ as

$$y^{\ 2)}\ t) = \ 1+\gamma t)\ 1-\gamma)^{t}.x\ 0) + \sum_{v=1}^{t}\ t-v+1).\gamma^{2}\ 1-\gamma)^{t-v}.x(v). \tag{9}$$

*Eq. (9)* can be used to induce over-smoothing of orders $3, 4, \ldots, n$.

## 4 | Smoothing with Piecewise Quadratic Fuzzy Data

Consider the smoothing problem with piecewise quadratic fuzzy data. Assume that the number $x(v)$ in the data is replaced by the corresponding close intervals approximation as

$$X\ v) = [X_{\alpha}^{-}\ v), X_{\alpha}^{+}\ v)] = [x_{1}\ v), x_{2}(v)]. \tag{10}$$

It is clear that a complex weighting process used in exponential smoothing preserves the monotonicity for the close intervals approximation. Therefore, the smoothed fuzzy sequence can be written using *Eqs. (8)* and *(9)* as

$$Y\ t) = \ 1-\gamma)^{t+1}X(0)(+)\sum_{v=0}^{t}\gamma\ 1-\gamma)^{t-v}.X\ v). \tag{11}$$

Or in the recursive form, *Eq. (11)* becomes

$$Y\ v) = [Y_{\alpha}^{-}\ v), Y_{\alpha}^{+}\ v)] = [y_{1}\ v), y_{2}(v)]. \tag{12}$$

## 5 | Numerical Examples

**Example 1.** Let $\gamma = 0.8$, and using the *Eq. (6)*, $x(v)$ and $y(v)$ can be calculated for $v = 1,2,\dots,7$ as

**Table 1. Calculations of x(v) and y(v) with non-fuzzy data.**

| x(v) | y(v) |
|---|---|
| $y(0) = 7$ | $y(0) = 7$ |
| $x(1) = 6$ | $y(1) = 0.8 \times 6 + 0.2 \times 7 = 6.2$ |
| $x(2) = 4$ | $y(2) = 0.8 \times 4 + 0.2 \times 6.2 = 4.44$ |
| $x(3) = 7$ | $y(3) = 0.8 \times 7 + -0.2 \times 4.44 = 6.488$ |
| $x(4) = 9$ | $y(4) = 0.8 \times 9 + 0.2 \times 6.488 = 8.4976$ |
| $x(5) = 13$ | $y(5) = 0.8 \times 13 + 0.2 \times 8.4976 = 12.09952$ |
| $x(6) = 8$ | $y(6) = 0.8 \times 8 + 0.2 \times 12.09952 = 8.819904$ |
| $x(7) = 12$ | $y(7) = 0.8 \times 12 + 0.2 \times 8.819904 = 11.3639808$ |

**Example 2.** Assume again $\gamma = 0.8$ and let the input sequence of PQF be given by

$$X\ 0) = [x_1\ 0), x_2\ 0)] = [6,9],$$
$$X\ 1) = [x_1\ 1), x_2\ 1)] = [5,8],$$
$$X\ 2) = [x_1\ 2), x_2\ 2)] = [3,4],$$
$$X\ 3) = [x_1\ 3), x_2\ 3)] = [9,9] = 9,$$
$$X\ 4) = [x_1\ 4), x_2\ 4)] = [6,8].$$

By using the *Recursive Expression (8)*, the lower and upper bounds of the smoothed (filtered) data can be computed as

**Table 2. Computations of the lower bound $y_1(t)$ corresponding to $x_1\ t)$.**

| $x_1(t)$ | $y_1(t)$ |
|---|---|
| $x_1(0) = 6$ | $y_1(0) = 6$ |
| $x_1(1) = 5$ | $y_1(1) = 0.8 \times 5 + 0.2 \times 6 = 5.2$ |
| $x_1(2) = 3$ | $y_1(2) = 0.8 \times 3 + 0.2 \times 5.2 = 3.44$ |
| $x_1(3) = 9$ | $y_1(3) = 0.8 \times 9 + 0.2 \times 3.44 = 7.888$ |
| $x_1(4) = 6$ | $y_1(4) = 0.8 \times 6 + 0.2 \times 7.888 = 6.3776$ |

**Table 3. Computations of the upper bound $y_2(t)$ corresponding to $x_2\ t)$.**

| $x_2(t)$ | $y_2(t)$ |
|---|---|
| $x_2(0) = 9$ | $y_2(0) = 9$ |
| $x_2(1) = 8$ | $y_2(1) = 0.8 \times 8 + 0.2 \times 9 = 8.2$ |
| $x_2(2) = 4$ | $y_2(2) = 0.8 \times 4 + 0.2 \times 8.2 = 4.84$ |
| $x_2(3) = 9$ | $y_2(3) = 0.8 \times 9 + 0.2 \times 4.84 = 8.168$ |
| $x_2(4) = 8$ | $y_2(4) = 0.8 \times 8 + 0.2 \times 8.168 = 8.0336$ |

Thus, the fuzzy smoothed estimates as

$$Y\ 0) = [6,9],\quad Y\ 1) = [5.2,8.2],\quad Y\ 2) = [3.44,4.84],$$
$$Y\ 3) = [7.888,8.168],\quad Y\ 4) = [6.3776,8.0336].$$

**Example 3.** Let the fuzzy time series given by PQFs as

$$X\ 0) = \ 3,5,7,8,9), X\ 1) = \ 4,4,5,5,5), X\ 2) = \ 11,11,12,13,14),$$
$$X\ 3) = \ 6,7,8,8,8), X\ 4) = \ 7,7,7,7,7)\ , X\ 5) = \ 5,5,6,7,9).$$

Let us define:

Input PQF time series: $X\ v) = \big(x_1\ v), x_2\ v), x_3\ v), x_4\ v), x_5\ v)\big)$.

Output (smoothed) PQF time series: $Y\ v) = \left(y_1\ v), y_2\ v), y_3\ v), y_4\ v), y_5\ v)\right)$.

Assuming that $\gamma = 0.7$ and using the recursive *Eq. (8)*, we obtain

**Table 4. Computations of $y_1(t)$ corresponding to $x_1$ t).**

| $x_1(t)$ | $y_1(t)$ |
|---|---|
| $x_1(0) = 3$ | $y_1(0) = 3$ |
| $x_1(1) = 4$ | $y_1(1) = 0.7 \times 4 + 0.3 \times 3 = 3.7$ |
| $x_1(2) = 11$ | $y_1(2) = 0.7 \times 11 + 0.3 \times 3.7 = 8.81$ |
| $x_1(3) = 6$ | $y_1(3) = 0.7 \times 6 + 0.3 \times 8.81 = 6.843$ |
| $x_1(4) = 7$ | $y_1(4) = 0.7 \times 7 + 0.3 \times 6.843 = 6.9529$ |
| $x_1(5) = 5$ | $y_1(5) = 0.7 \times 5 + 0.3 \times 6.9529 = 5.58587$ |

**Table 5. Computations of $y_2(t)$ corresponding to $x_2$ t).**

| $x_2(t)$ | $y_2(t)$ |
|---|---|
| $x_2(0) = 5$ | $y_2(0) = 5$ |
| $x_2(1) = 4$ | $y_2(1) = 0.7 \times 4 + 0.3 \times 5 = 4.3$ |
| $x_2(2) = 11$ | $y_2(2) = 0.7 \times 11 + 0.3 \times 4.3 = 8.99$ |
| $x_2(3) = 7$ | $y_2(3) = 0.7 \times 7 + 0.3 \times 8.99 = 7.597$ |
| $x_2(4) = 7$ | $y_2(4) = 0.7 \times 7 + 0.3 \times 7.597 = 7.1791$ |
| $x_2(5) = 5$ | $y_2(5) = 0.7 \times 5 + 0.3 \times 7.1791 = 5.65373$ |

**Table 6. Computations of $y_3(t)$ corresponding to $x_3$ t).**

| $x_3(t)$ | $y_3(t)$ |
|---|---|
| $x_3(0) = 7$ | $y_3(0) = 7$ |
| $x_3(1) = 5$ | $y_3(1) = 0.7 \times 5 + 0.3 \times 7 = 5.6$ |
| $x_3(2) = 12$ | $y_3(2) = 0.7 \times 12 + 0.3 \times 5.6 = 10.08$ |
| $x_3(3) = 8$ | $y_3(3) = 0.7 \times 8 + 0.3 \times 10.08 = 8.624$ |
| $x_3(4) = 7$ | $y_3(4) = 0.7 \times 7 + 0.3 \times 8.623 = 7.4872$ |
| $x_3(5) = 6$ | $y_3(5) = 0.7 \times 6 + 0.3 \times 7.4872 = 6.44616$ |

**Table 7. Computations of the upper bound $y_4(t)$ corresponding to $x_4$ t).**

| $x_4(t)$ | $y_4(t)$ |
|---|---|
| $x_4(0) = 8$ | $y_4(0) = 8$ |
| $x_4(1) = 5$ | $y_4(1) = 0.7 \times 5 + 0.3 \times 8 = 5.9$ |
| $x_4(2) = 13$ | $y_4(2) = 0.7 \times 13 + 0.3 \times 5.9 = 10.87$ |
| $x_4(3) = 8$ | $y_4(3) = 0.7 \times 8 + 0.3 \times 10.87 = 6.161$ |
| $x_4(4) = 7$ | $y_4(4) = 0.7 \times 7 + 0.3 \times 6.161 = 8.5966$ |
| $x_4(5) = 7$ | $y_4(5) = 0.7 \times 7 + 0.3 \times 8.5966 = 6.57898$ |

**Table 8. Computations of the upper bound $y_5(t)$ corresponding to $x_5$ t).**

| $x_5(t)$ | $y_5(t)$ |
|---|---|
| $x_5(0) = 9$ | $y_5(0) = 9$ |
| $x_5(1) = 5$ | $y_5(1) = 0.7 \times 5 + 0.3 \times 9 = 6.2$ |
| $x_5(2) = 14$ | $y_5(2) = 0.7 \times 14 + 0.3 \times 6.2 = 11.66$ |
| $x_5(3) = 8$ | $y_5(3) = 0.7 \times 8 + 0.3 \times 11.66 = 9.098$ |
| $x_5(4) = 7$ | $y_5(4) = 0.7 \times 7 + 0.2 \times 9.098 = 7.6294$ |
| $x_5(5) = 9$ | $y_5(5) = 0.7 \times 9 + 0.3 \times 7.6294 = 8.58882$ |

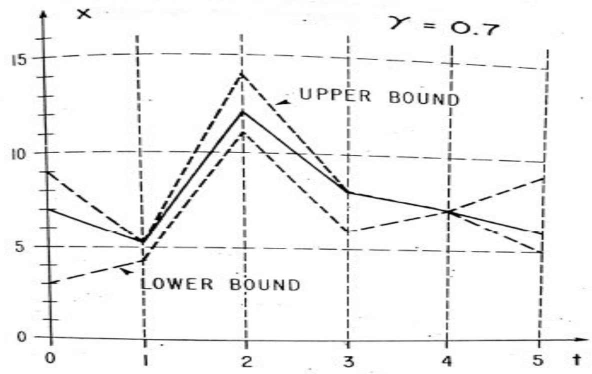


**Fig. 3. Input fuzzy time series X(t) (Example 3).**

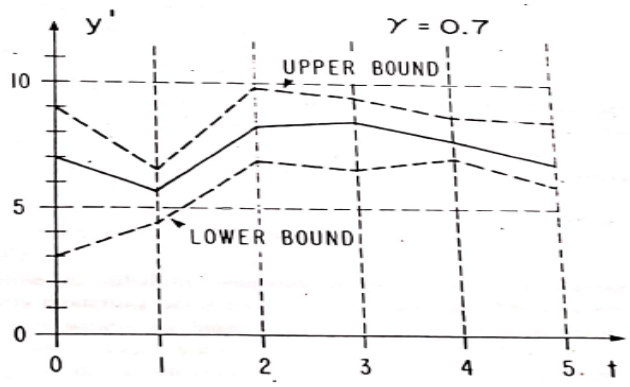


**Fig. 4. First-ordered smoothed time series Y t) (Example 3).**

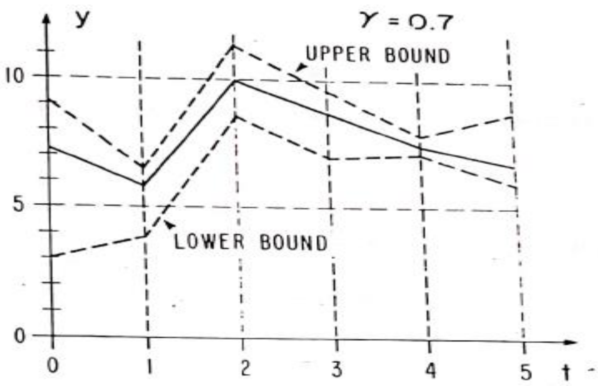


**Fig. 5. Second ordered smoothed time series Y′(t) (Example 3).**

## 6 | Conclusions and Future Works

This paper clearly shows that using PQFNs gives a more unambiguous indication of the effect of fuzzy smoothing (filtering). We have not addressed some critical questions which may arise when the smoothing is applied to a fuzzy time series. These include, for instance, the choice of the smoothing parameter $\gamma$, the forecasting of several periods after smoothing, and the adaptive filtering, etc. We aimed to explain that in smoothing and extrapolation problems, fuzzy numbers are just a generalization of ordinary numbers.

Most of the concepts of filtering and estimation, which are well-known in statistical theory, can be extended to fuzzy filtering and estimation with fuzzy time sequences. Future work will be focused on fuzzy optimal estimation, adaptive estimation, and filtering. Future work may include extending this study to other fuzzy-like structures (i.e., interval-valued fuzzy set, Neutrosophic set, Pythagorean fuzzy set, Spherical fuzzy set, etc., with more discussion and suggestive comments

## Conflicts of Interest

The authors declare no conflicts of interest.

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