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Developed a New Fuzzy Approach for Solving Two-Machine Flow Shop Scheduling Problems under Fuzziness

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Abstract

The current study investigates a two-machine Flow Shop Scheduling (FSS) problem with piecewise quadratic fuzzy processing time. It is illogical to consider that the processing time is exact but uncertain because it varies due to human factors. One of the most popular approximate intervals, namely, close interval approximation for the Piecewise Quadratic Fuzzy Number (PQFN), is introduced. A solution method with the help of Johnson's algorithm [1], the close interval approximation of PQFNs, and the modified McCahon and Lee's algorithm [2] is developed to determine the minimization of the expected makespan. Numerical experimentation is performed to demonstrate the effectiveness of the suggested methodology.

Keywords: Optimization problems, Production scheduling, Two-machine flow shops problem, Piecewise quadratic fuzzy numbers, Decision making, Close interval approximation, Expected makespan, Optimal sequence.

1 | Introduction

In many practical-life applications, a Flow Shop Scheduling (FSS) problem occurs [3]. Agrawal et al. [4] encountered three FS device scheduling problems to minimize a machine's completion time. Vahedi–Nouri et al. [5] presented a broader version of the FS strategy to minimize the average flow rate. To optimize the publication time, Ren et al. [6] explored the topic of FS programming. Laribi et al. [7] introduced a mathematical model for two FS-limited machines that address FS time reduction problems where renewables are unconstrained. Yazdani and Naderi [8] considered the scheduling problem with no-idle hybrid FSS and applied mixed integer linear programming to formulate the problem. Qu et al. [9] designed a flower pollination algorithm based on the hormone modulation mechanism for the no-wait FSS problem.

In literature, Zadeh [10] proposed the philosophy of fuzzy sets. Bellman and Zadeh [11] and then Dubois and Prade [12] spread the expansion of algebraic operations into fuzzy numbers by a fuzzy hypothesis of real numbers. Panda and Pal [13] developed the theory of pentagonal andantes fuzzy

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numbers. Many scientists were involved in the problem of FS preparation [14], [15]. Yao and Lin [16] introduced a method containing a variety of statistical and fuzzy flow-shop sets. Hejari et al. [17] enhanced McCahon and Lee's algorithm [2] for work sequence with a fuzzy processing time. For a smooth processing time, Ishibuchi and Lee [18] established the FS scheduling problem. Temiz and Erol [19] also employed the fuzzy scheduling principle of the FS, updated the branch-and-bound of Ignall and Schrage [20] and reassigned the Ignall and Schrage [20] algorithm to three FS frame problems. Gupta et al. [21] applied a heuristic algorithm to minimize the rental cost of the machines for a specially structured three-stage FSS. González-Neira et al. [22] overviewed the FS scheduling problem under uncertainties. Komaki et al. [23] conducted a joint survey of their templates for assembly FSS. Janaki and Mohamed Ismail [24] explored the interval modelling work using an efficient and heuristic make-up approach. Khalifa [25] analyzed the single-machine preparation issue in a fizzy dates setting. Khalifa et al. [26] proposed a fuzzy binding approach for solving constrained FSS problems without converting the problem into its crisp. Alhabi and Khalifa [27] proposed an approach for solving the FSS with a pentagonal processing time of jobs. Many researchers have introduced approaches for solving FSP (for instance, Alburikan et al. [28], Ren et al. [29], Wang et al. [30], [31], Jemmai and Hidri [32] and Koulamas and Kyparisis [33]).

This study investigates the problem of two machine FSS in a fuzzy environment. This study aims to find a job sequence with minimum makespan. An algorithm introduced by McCahon and Lee [2] can be applied with some modifications, that is, by using the close interval approximation of PQF processing times. This modification leads to the performance improvement of the scheduling problem.

The rest of the paper is outlined as follows:

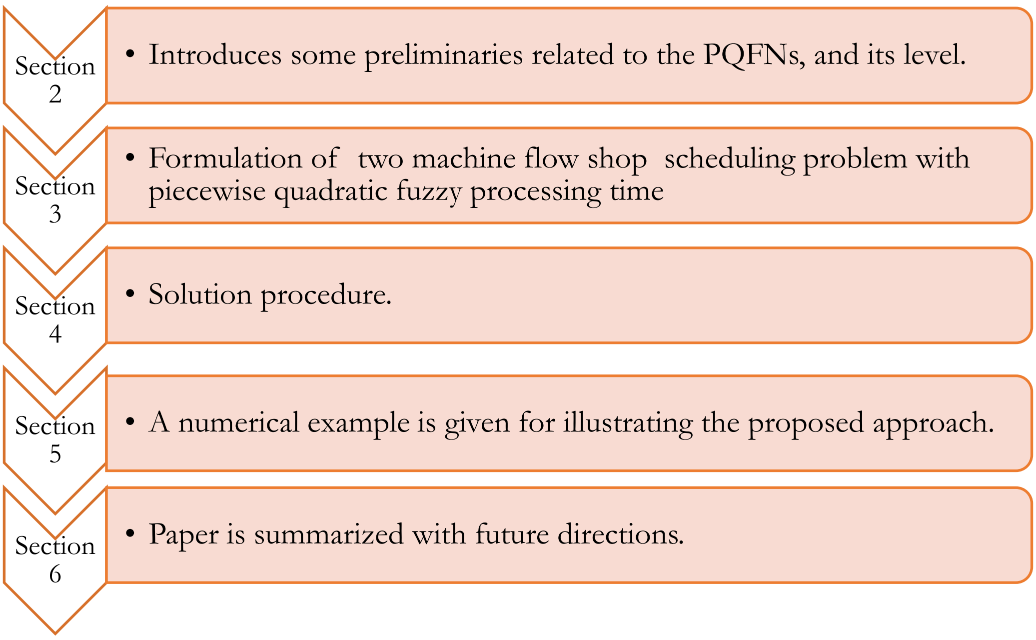

Fig. 1. Paper structure.

2 | Preliminaries

Definition 1 ([10]). Fuzzy number: A fuzzy number A is a fuzzy set with a membership function defined as $\pi_A(x): \Re \rightarrow [0,1]$, and satisfies:

I. A is fuzzy convex, i.e., $\pi_A(\delta x + (1-\delta) y) \geq \min\{\pi_A(x), \pi_A(y)\}$; $for\ all\ x, y \in \Re; 0 \leq \delta \leq 1$;

II. A is normal, i.e., $\exists\, x_0 \in \Re$ for which $\pi_A(x_0) = 1$.

III. $Supp\left(A\right) = \{x \in \Re: \pi_A(x) > 0\}$ is the support of A.

IV. $\pi_A(x)$ is an upper semi-continuous (i.e., for each $\alpha \in (0,1)$, the $\alpha -$ cut set $A_\alpha = \{x \in \Re: \pi_A \geq \alpha\}$ is closed.

Definition 2 ([34]). A Piecewise Quadratic Fuzzy Number (PQFN) is denoted by $A_{PQ} = \ a_1, a_2, a_3, a_4, a_5)$, where $a_1 \leq \ a_2 \leq a_3 \leq \ a_4 \leq \ a_5$ are real numbers and is defined by if its membership function $\mu_{a_{PQ}}$ is given by (see *Fig. 2*)

$$\mu_{\widetilde{A}_{PQ}} = \begin{cases} 0, & x < a_1, \\ \dfrac{1}{2}\dfrac{1}{a_2 - a_1)^2}\ x - a_1)^2, & a_1 \leq x \leq a_2, \\ \dfrac{1}{2}\dfrac{1}{a_3 - a_2)^2}\ x - a_3)^2 + 1, & a_2 \leq x \leq a_3, \\ \dfrac{1}{2}\dfrac{1}{a_4 - a_3)^2}\ x - a_3)^2 + 1, & a_3 \leq x \leq a_4, \\ \dfrac{1}{2}\dfrac{1}{a_5 - a_4)^2}\ x - a_5)^2, & a_4 \leq x \leq a_5, \\ 0, & x > a_5. \end{cases}$$

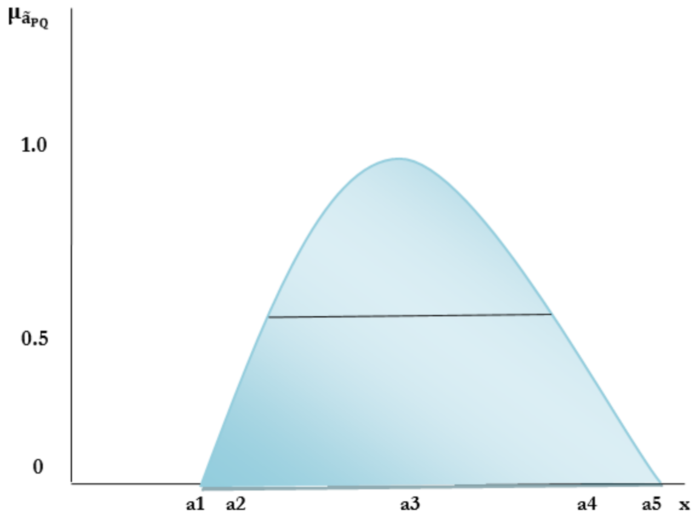

Fig. 2. Graphical representation of PQFN.

Definition 3 ([34]). Let $A_{PQ} = \ a_1, a_2, a_3, a_4, a_5)$ and $B_{PQ} = \ b_1, b_2, b_3, b_4, b_5)$ be two PQFNs. The arithmetic operations on A_{PQ} and B_{PQ} are:

I. Addition: $A_{PQ}(+)B_{PQ} = \ a_1 + b_1, a_2 + \ b_2, a_3 + \ b_3, a_4 + b_4, a_5 + b_5)$.

II. Subtraction: $A_{PQ}(-)B_{PQ} = \ a_1 + b_5, a_2 + \ b_4, a_3 + \ b_3, a_4 + b_2, a_5 + b_1)$

III. Scalar multiplication: $kA_{PQ} = \begin{cases} \ k\ a_1, k\ a_2, k\ a_3, k\ a_4, k\ a_5), k > 0, \\ \ k\ a_5, k\ a_4, k\ a_3, k\ a_2, k\ a_1), k < 0. \end{cases}$

IV. Maximum: $A_{PQ}\ \vee)B_{PQ} = \ a_1 \vee b_1, a_2 \vee \ b_2, a_3 \vee \ b_3, a_4 \vee b_4, a_5 \vee b_5)$.

V. Minimum: $A_{PQ}\ \wedge)B_{PQ} = \ a_1 \wedge b_1, a_2 \wedge \ b_2, a_3 \wedge \ b_3, a_4 \wedge b_4, a_5 \wedge b_5)$.

VI. Scalar multiplication: $k\bigodot A_{PQ} = \begin{cases} \ k\ a_1, k\ a_2, k\ a_3, k\ a_4, k\ a_5), k > 0; \\ \ k\ a_5, k\ a_4, k\ a_3, k\ a_2, k\ a_1), k < 0. \end{cases}$

Definition 4 ([34]). An interval approximation $[A] = \left[a_\alpha^L, a_\alpha^U\right]$ of a PQFN A is called closed interval approximation if $a_\alpha^L = inf\left\{x \in \mathbb{R}\text{: } \mu_A \geq 0.5\right\},\ and\ a_\alpha^U = sup\left\{x \in \mathbb{R}\text{: } \mu_A \geq 0.5\right\}$.

Definition 5 ([34]). If $[A] = \left[a_\alpha^L, a_\alpha^U\right]$ is the close interval approximation of PQFN, the associated ordinary number of $[A]$ is defined as $A = \frac{a_\alpha^L + a_\alpha^U}{2}$.

Definition 6 ([34]). Let $[A]=\left[a_{\alpha}^{L},a_{\alpha}^{U}\right]$, and $[B]=\left[b_{\alpha}^{L},b_{\alpha}^{U}\right]$ be two interval approximations of PQFN, then the algebraic operations are:

I. Addition: $[A](+)[B]=\left[a_{\alpha}^{L}+b_{\alpha}^{L},a_{\alpha}^{U}+b_{\alpha}^{U}\right]$.

II. Subtraction: $[A]-)[B]=\left[a_{\alpha}^{L}-b_{\alpha}^{U},a_{\alpha}^{U}-b_{\alpha}^{L}\right]$.

III. Scalar multiplication: $\alpha\,[A]=\begin{cases}\left[\alpha\, a_{\alpha}^{L},\ \alpha\, a_{\alpha}^{U}\right],\alpha>0,\\ \left[\ \alpha\, a_{\alpha}^{U},\alpha\, a_{\alpha}^{L}\right],\alpha<0.\end{cases}$

IV. Multiplication: $[A](\times)[B]=\left[\frac{a_{\alpha}^{U}b_{\alpha}^{L}+a_{\alpha}^{L}b_{\alpha}^{U}}{2},\frac{a_{\alpha}^{L}b_{\alpha}^{L}+a_{\alpha}^{U}b_{\alpha}^{U}}{2}\right]$.

V. Division: $[A]\div)[B]=\begin{cases}\left[2\left(\frac{a_{\alpha}^{L}}{b_{\alpha}^{L}+b_{\alpha}^{U}}\right),2\left(\frac{a_{\alpha}^{U}}{b_{\alpha}^{L}+b_{\alpha}^{U}}\right)\right],[B]>0,b_{\alpha}^{L}+b_{\alpha}^{U}\neq 0,\\ \left[2\left(\frac{a_{\alpha}^{U}}{b_{\alpha}^{L}+b_{\alpha}^{U}}\right),2\left(\frac{a_{\alpha}^{L}}{b_{\alpha}^{L}+b_{\alpha}^{U}}\right)\right],[B]<0,b_{\alpha}^{L}+b_{\alpha}^{U}\neq 0.\end{cases}$

VI. Minimum: $[A]\wedge)[B]=[A](\wedge)[B]=\left[a_{\alpha}^{L},a_{\alpha}^{U}\right]\ \wedge)\left[b_{\alpha}^{L},b_{\alpha}^{U}\right]=\left[a_{\alpha}^{L}\wedge b_{\alpha}^{L},a_{\alpha}^{U}\wedge b_{\alpha}^{U}\right]$.

VII. Maximum: $[A](\wedge)[B]=[A](\vee)[B]=\left[a_{\alpha}^{L},a_{\alpha}^{U}\right]\ \vee)\left[b_{\alpha}^{L},b_{\alpha}^{U}\right]=\left[a_{\alpha}^{L}\vee b_{\alpha}^{L},a_{\alpha}^{U}\vee b_{\alpha}^{U}\right]$.

VIII. The order relations:

- $[A]\lesssim)[B]$ *if* $a_{\alpha}^{L}\leq b_{\alpha}^{L}$ *and* $a_{\alpha}^{U}\leq b_{\alpha}^{U}$ *or* $a_{\alpha}^{L}+a_{\alpha}^{U}\leq b_{\alpha}^{L}+b_{\alpha}^{U}$.
- $[A]$ *is preferred to* $[B]$ *if and only if* $a_{\alpha}^{L}\geq b_{\alpha}^{L},a_{\alpha}^{U}\geq b_{\alpha}^{U}$.

3 | Problem Formulation

Consider n non-similar jobs that pass through m machines. A machine processes one job at a time and assumes that the order is not changed once the processing begins. Suppose that a simple FSS problem with jobs and only two machines. The jobs solution sequence is the permutation of $\overline{1,n}$, and $x_i, i=\overline{1,n}$ represent the job sequence. Let $\left(R_{PQ}\right)_{x_i,j}$ and $\left(C_{PQ}\right)_{x_i,j}$ be the piecewise quadratic fuzzy processing time and completion time of job x_i on machine $i=1,2$; respectively. To calculate the completion time for each job, let us apply the arithmetic operations of PQFNs, defined in *Definition 3*.

The problem is formulated mathematically as follows:

Determine a sequence of jobs to obtain minimum makespan C_{PQ} such that

$$\left(\widetilde{\mathrm{C}}_{\mathrm{PQ}}\right)_{\mathrm{x_i,j}}=\left(\mathrm{R}_{\mathrm{PQ}}\right)_{\mathrm{x_i,j}}. \tag{1}$$

$$\left(\widetilde{\mathrm{C}}_{\mathrm{PQ}}\right)_{\mathrm{x_i,1}}=\left(\widetilde{\mathrm{C}}_{\mathrm{PQ}}\right)_{\mathrm{x_{i-1},1}}\bigoplus\left(\mathrm{R}_{\mathrm{PQ}}\right)_{\mathrm{x_i,1}},\text{for } \mathrm{i}=\overline{1,\mathrm{n}}. \tag{2}$$

$$\left(\widetilde{\mathrm{C}}_{\mathrm{PQ}}\right)_{\mathrm{x_1,j}}=\left(\widetilde{\mathrm{C}}_{\mathrm{PQ}}\right)_{\mathrm{x_1,j-1}}\bigoplus\left(\mathrm{R}_{\mathrm{PQ}}\right)_{\mathrm{x_1,j}},\text{for } \mathrm{j}=\overline{2,\mathrm{m}}. \tag{3}$$

$$\left(\widetilde{\mathrm{C}}_{\mathrm{PQ}}\right)_{\mathrm{x_i,j}}=\left(\left(\widetilde{\mathrm{C}}_{\mathrm{PQ}}\right)_{\mathrm{x_{i-1},j}}\vee\left(\widetilde{\mathrm{C}}_{\mathrm{PQ}}\right)_{\mathrm{x_i,j-1}}\right)\oplus\left(\mathrm{R}_{\mathrm{PQ}}\right)_{\mathrm{x_i,j}},\text{for } \mathrm{i}=\overline{1,\mathrm{n}};\mathrm{j}=\overline{1,\mathrm{m}}. \tag{4}$$

Referring to the representation theorem that states any fuzzy set R are decomposed into a series of its $\alpha-$ cut sets (i., $R_{PQ}=\bigcup_{\alpha\in[0,1]}\alpha\, R_{PQ}^{\alpha}$. Conversely, any fuzzy set may be derived from a family of nested sets, providing that if $\alpha_1>\alpha_2, R_{PQ}^{\alpha_1}\subset R_{PQ}^{\alpha_2}$. It is clear that this theorem implies that the formulation of the problem in the fuzzy set framework can be solved by converting these fuzzy sets into their corresponding $\alpha-$ cuts families. In addition, for all $\alpha-$ cuts, the derived results can be merged to obtain a solution for the problem with fuzzy sets.

Fuzzy makespan C_{PQ} can be calculated by considering all of its $\alpha-$ cuts (i., $C_{PQ}^{\alpha},0\leq\alpha\leq 1$). To calculate a makespan at $\alpha-$ cuts, we use $C_{PQ}=\ c_1,c_2,c_3,c_4,c_5)$, where c_1 and c_5 are the lower and upper bounds of the membership function C_{PQ}; respectively, and c_3 is the modal point.

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The fuzzy makespan C_{PQ} obtained by McCahon and Lee's algorithm [2] has been evaluated by using the Yager index as

$$Y(\widetilde{C}_{PQ}) = \frac{1}{2}\int_0^1 C_L^\alpha + C_U^\alpha)d\alpha. \tag{5}$$

Now, let us approximate a fuzzy number by a deterministic number. Suppose that A_{PQ} and B_{PQ} be two PQFNs, with $\alpha -$ cuts $A_{PQ}^\alpha = [A_L^\alpha, A_U^\alpha]$ and $B_{PQ}^\alpha = [B_L^\alpha, B_U^\alpha]$; respectively. Then, the distance between $\tilde{A}_{PQ}$ and B_{PQ} is

$$D(\widetilde{A}_{PQ}, B_{PQ}) = \left(\int_0^1 A_L^\alpha - B_L^\alpha)^2 d\alpha + \int_0^1 A_U^\alpha - B_U^\alpha)^2 d\alpha\right)^{1/2}. \tag{6}$$

We aim to find a close interval $C_D(A_{PQ})$ that is nearest to A_{PQ} with respect to metric D (i.e, $\left(C_D(A_{PQ})\right)^\alpha =$ $[C_L, C_U]$. Now, we want to

$$\text{minimize } D(\widetilde{A}_{PQ}, C_D(\widetilde{A}_{PQ})) = \left({}_0^1 A_L^\alpha - C_L)^2 d\alpha + {}_0^1 A_U^\alpha - C_U)^2 d\alpha\right)^{1/2},$$
$$\text{Subject to } C_L \text{ and } C_U. \tag{7}$$

To minimize $D(A_{PQ}, C_D(A_{PQ}))$, it is sufficient to minimize $D'\ C_L,\ C_U) = \left(D^2(A_{PQ}, C_D(A_{PQ}))\right)$ as follows:

$$\frac{\partial D'\ C_L, C_U)}{\partial C_L} = -2\int_0^1 A_L^\alpha - C_L)^2 d\alpha = -2\int_0^1 A_L^\alpha d\alpha + 2C_L = 0. \tag{8}$$

$$\frac{\partial D'\ C_L, C_U)}{\partial\ C_U} = -2\int_0^1 A_U^\alpha - C_U)^2 d\alpha = -2\int_0^1 A_U^\alpha d\alpha + 2C_U = 0. \tag{9}$$

By solving *Eqs. (8)* and *(9)*, we have

$$C_L^* = {}_0^1 A_U^\alpha d\alpha, \text{ and } C_U^* = {}_0^1 A_U^\alpha d\alpha. \tag{10}$$

The Hessian matrix is given by

$$H\ C_L^*, C_U^*) = \frac{\partial^2}{\partial C_L^2} D'\ C_L^*, C_U^*)\frac{\partial^2}{\partial C_U^2} D'\ C_L^*, C_U^*) - \left(\frac{\partial^2}{\partial C_L \partial C_U} D'\ C_L^*, C_U^*)\right)^2 = 4 > 0.$$

Hence, $D'(A_{PQ}, C_D(A_{PQ}))$ is the required global solution.

Therefore, the close interval

$$C_D(\widetilde{A}_{PQ}) = \left[C_L = {}_0^1 A_U^\alpha d\alpha,\ C_U = {}_0^1 A_U^\alpha d\alpha\right], \tag{11}$$

is the close interval approximation for a fuzzy number A_{PQ} with respect to metric D.

Let $A_{PQ} =\ a_1, a_2, a_3, a_4, a_5)$ be a PQFN. The interval of confidence at level α, is defined as

$$\widetilde{A}_{PQ}^\alpha = [a_1 + 2\ a_2 - a_1)\alpha, a_5 - 2\ a_5 - a_4)\ \alpha] = [A_L^\alpha, A_U^\alpha].$$

By the close interval approximation, C_L and C_U are defined as

$$C_L = \int_0^1 A^{\alpha}_L d\alpha = \int_0^1 a_1 + 2\, a_2 - a_1)\alpha)d\alpha = a_2. \tag{12}$$

$$C_U = \int_0^1 A^{\alpha}_U d\alpha = \int_0^1 (a_5 - 2(a_5 - a_4)\, \alpha)d\alpha = a_{4.} \tag{13}$$

4 | A Solution Method Based on the Close Interval Approximation of PQFN

A solution method steps for solving the problem can be summarized in the following steps:

Step 1. Consider piecewise quadratic fuzzy processing times of two jobs on two machines.

Step 2. Apply the close interval approximation for each PQF processing time.

Step 3. Calculate the minimum and the maximum of any two close interval approximations, the associated ordinary number of each interval and then use Johnson's algorithm [1] to find an optimal job sequence.

Step 4. Calculate the makespan based on the optimal job sequences obtained from *Step 3* after converting all the PQF processing times to the close interval approximation. Then, the makespan can be calculated as a close interval approximation as follows:

Using the close interval approximation, the problem can be defined as follows:

Let $[R_{x_i,j}] = [R^L_{\alpha}, R^U_{\alpha}], i = \overline{1,n}; j = 1,2$ and $[C_{x_i,j}] = [C^L_{\alpha},\ C^U_{\alpha}], i = \overline{1,n}, j = 1,2$ be the close interval approximation of processing time and completion time of job x_i on machine $i = 1,2$; respectively.

Find a sequence of jobs to achieve minimum makespan $[C_{x_i,2}]$ such that

$$[C_{x_i,1}] = [R_{x_i,1}]. \tag{14}$$

$$[C_{x_i,1}] = [C_{x_{i-1},1}]\ +)[R_{x_i,j}]\ i = \overline{1,n}. \tag{15}$$

$$[C_{x_i,2}] = [C_{x_i,1}]\ +)[R_{x_i,2}]. \tag{16}$$

$$[C_{x_i,2}] = ([C_{x_{i-2},2}] \vee [C_{x_i,1}])\ +)[R_{x_i,2}], i = \overline{1,n}. \tag{17}$$

The makespan is a close interval approximation, and thus, the associated ordinary number of the makespan is defined as follows:

$$\widehat{C}_{x_i,2} = \frac{C^L_{\alpha} + C^U_{\alpha}}{2}. \tag{18}$$

5 | Numerical Experimentation

Example 1. Consider a simple two-machine FSS problem with only two jobs. *Table 1* introduces the processing times of two machines. M_1 and M_2 represent machines 1 and 2, respectively.

Table 1. Two-jobs, two-machine FSS problem with PQF processing times.

Machine \ Job	x_1	x_2
M_1	(1, 2, 2, 18, 36)	(5, 6, 8, 16, 23)
M_2	(21, 22, 25, 26, 28)	(9, 10, 12, 13, 16)

The solution by McCahon and Lee [2]:

Step 1. Let us apply the Generalized Mean Values (GMVs) for the jobs as follows (*Table 2* to *Table 6*).

Table 2. Two-jobs, two-machine FSS problem with GMVs for processing times.

Machine \ Job	x_1	x_2
M_1	10	11
M_2	24	11.5

Table 3. McCahon and Lee [2] solution: sequence x_2, x_1.

Machine \ Job	x_2	x_1
M_1	(5, 6, 8, 16, 23)	(6, 8, 10, 34, 59)
M_2	(14, 16, 20, 29, 39)	(35, 38, 45, 53, 67)

Table 4. Optimal solution: sequence x_1, x_2.

Machine \ Job	x_1	x_2
M_1	(1, 2, 2, 18, 36)	(6, 8, 10, 34, 59)
M_2	(21, 22, 25, 26, 28)	(31, 34, 39, 57, 80)

Step 2. Convert all the PQF processing times into its close interval approximation as follows:

Table 5. Two-job two-machine FSS problem with close interval approximation processing times.

Machine \ Job	x_1	x_2
M_1	[2, 18]	[6, 16]
M_2	[22, 26]	[10, 13]

Step 3. Apply Johnson's algorithm [1] to determine the optimal sequence as follows:

$$[2,18] \wedge)[10,13] = [2,13] <)[22,26] \wedge)[6,16] = [6,16]. \tag{18}$$

Thus, the job sequence should be $x_1,\ x_2$ (as illustrated in *Table 6*).

Step 4. Determine the makespan. The makespan obtained is [34, 57], and the associated ordinary number is 45.5.

Table 6. Optimal solution (Sequence, x_1, x_2).

Machine \ Job	x_1	x_2
M_1	[2, 18]	[8, 34]
M_2	[24, 44]	[34, 57]

From *Table 6*, the results obtained are more satisfactory than those obtained by McCahon and Lee's algorithm [2].

Example 2. Consider two machine FSS problems with six jobs. *Table 1* introduces the processing times of two machines. M_1 and M_2 represent machines 1 and 2, respectively.

Table 7. FSS problem with PQF processing times.

Machine \ Job	1	2	3	4	5	6
M_1	(1, 6,14, 30, 40)	(20, 25, 33, 34,35)	(3, 5,7, 10, 19)	(18, 20, 24, 30,37)	(17,19, 21, 25, 32)	(13, 14,16, 20, 36)
M_2	(12,14,16, 17, 18)	(50,55, 60 62, 65)	(24,26,30, 35, 40)	(7, 9,10, 12, 15)	(8, 9, 11, 13, 14)	(3, 5, 8, 12, 20)

Table 8. Comparative study.

Solution Method	Sequence of Job	Makespan
McCahon and Lee's algorithm	$3 \to 2 \to 1 \to 5 \to 4 \to 6$	152
Full Search's solution	$3 \to 2 \to 1 \to 6 \to 5 \to 4$	145
Proposed solution method	$3 \to 2 \to 1 \to 6 \to 5 \to 4$	145

6 | Conclusion

In this paper, the FSS problem has been studied in a fuzzy environment. The concept of the close interval approximation of the PQFNs has been used to find a job permutation which minimizes the makespan. This operation avoids the information missing on fuzzy processing times. A comparative study has been achieved. In addition, the proposed method can be applied to different types of fuzzy numbers and in uncertain environments.

Availability of Data

No data backed up this study.

Conflicts of Interests

The authors do not have any conflicts of interest.

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